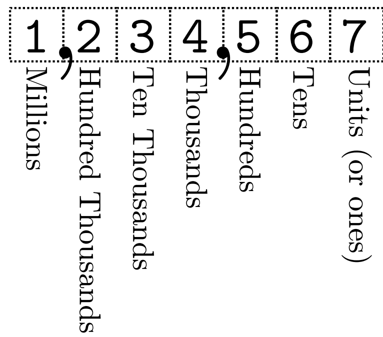
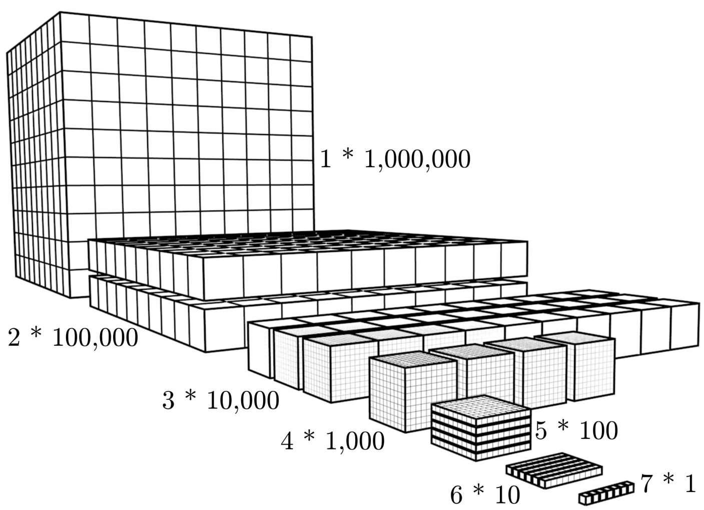
1 WHOLE NUMBERS

1.1 Positional notation

We use *positional notation* to represent numbers. Thus, while using only ten symbols (digits), we can write any whole number. Each position has a *place value* (1, 10, 100, etc). A number is written below with the places labeled.



This number (one million two hundred thirty four thousand five hundred sixty seven) can be visualized by arranging cubes. There are 7 copies of the smallest (unit) cube, 6 ten-unit rods, 5 hundred-unit plates, 4 thousand-unit cubes, etc.



1.2 Ordering whole numbers

Whole numbers have an order. As we count, the numbers grow larger. We are sometimes asked to compare two numbers and decide which is larger. We use the symbol "$>$" to mean "greater than" and the symbol "$<$" to mean "less than".

For example, which is larger, 201231 or 200982?

$$201231 > 200982$$

To determine which number is larger, you start with the highest non-zero place and move to lower places until one number has a smaller digit. The one with a smaller digit is smaller, and the other is larger.

Practice ordering whole numbers

1. Fill the box below with the proper symbol (either $>$ or $<$).

$$1823049 \;\square\; 239183$$

2. Fill the box below with the proper symbol (either $>$ or $<$).

$$10698 \;\square\; 10723$$

1.3 Rounding whole numbers

When asked to round a number at a specified place, you should determine what number is closest to the original but has zeros in lower places. You can find the result procedurally:

- If the digit to the right is either 5, 6, 7, 8, or 9, then the specified digit is incremented by 1.
- If a 9 is incremented, the 9 becomes 0 while incrementing the digit in the next higher place.
- All digits right of the digit specified are replaced with 0.

Practice rounding whole numbers

1. Round 4,256 at the hundreds place.

2. What is 8,913,489 rounded to the nearest thousand?

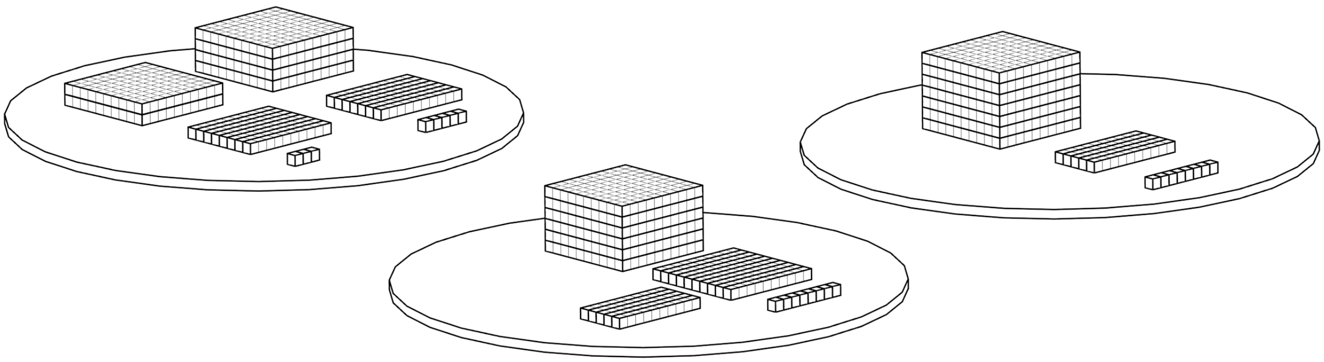
3. What is 99,999 rounded to the nearest ten?

4. Round 249 at the thousands place.

5. Barbara rounds her checking-account balance to the nearest hundred and gets \$4,300. What are the least and most whole-number balances that would round as stated?

1.4 Addition

Positional notation makes addition easy. For example, when 283 is added to 475 there are 6 hundreds, 15 tens, and 8 ones. The 15 tens can be understood as 1 hundred and 5 tens.



$$283 + 475 \quad = \quad 6 * 100 \ + \ 15 * 10 \ + \ 8 * 1 \quad = \quad 7 * 100 \ + \ 5 * 10 \ + \ 8 * 1 \quad = \quad 758$$

1.5 Approximate addition

Rounded numbers are easier to add. An adept arithmetician always double checks the calculator's result (which can be error-prone due to human error) with approximate mental arithmetic.

Practice approximate addition

1. Add 1,499,512 and 821,091 by first rounding at the ten-thousands place.

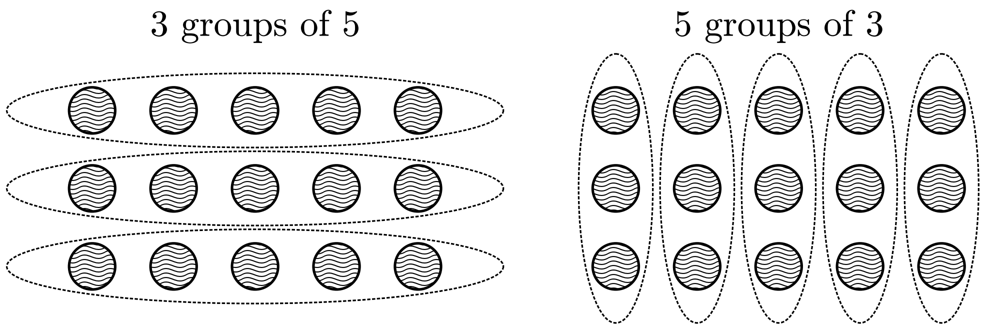
2. Add 1,499,512 and 821,091 by first rounding at the millions place.

3. First round 2250 and 150 to their nearest hundred, then add.

4. First round 2349 and 249 to their nearest hundred, then add.

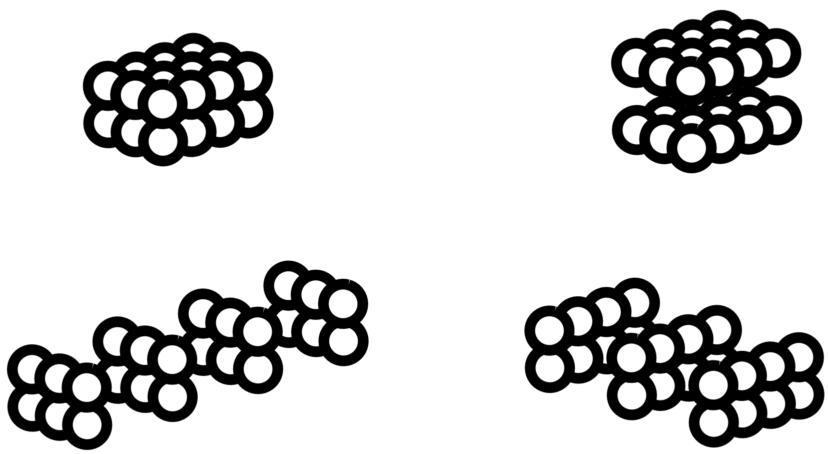
1.6 Multiplication

Multiplication is repeated addition. $3 * 5$ can be visualized as 3 groups of 5 objects or as 5 groups of 3 objects.



$$3 * 5 \; = \; 5 + 5 + 5 \; = \; 3 + 3 + 3 + 3 + 3 \; = \; 15$$

When we multiply numbers, the order of multiplication does not matter. For example, we can see that $2 * 3 * 4$ can be thought of as 2 groups of 3 groups of 4, 2 groups of 4 groups of 3, 3 groups of 2 groups of 4, 3 groups of 4 groups of 2, 4 groups of 2 groups of 3, or 4 groups of 3 groups of 2.



Also, it should be noted that multiplication is distributive over addition. For any values of a, b, and c, the following will be true.

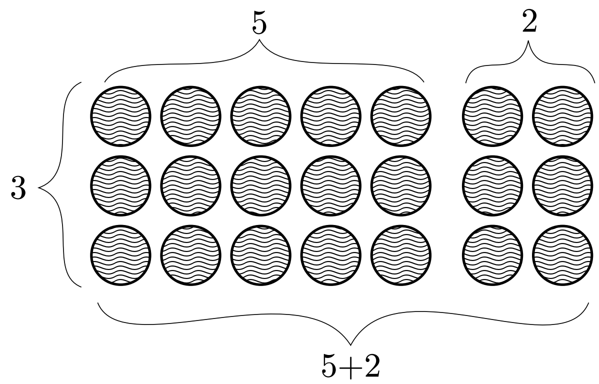
$$a * (b + c) = a * b + a * c$$

So,

$$11 * (8 + 9) \; = \; 11 * 8 + 11 * 9 \; = \; 88 + 99 \; = \; 187$$

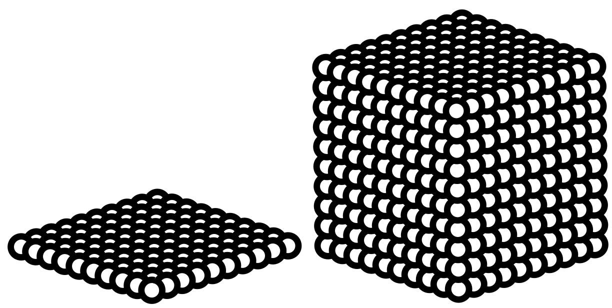
Of course, the above expression could also be evaluated by multiplying 11 by 17.

The distributive property is rather intuitive when multiplication is seen as rows and columns. For example, the following figure shows that $3 * (5 + 2) \; = \; 3 * 5 + 3 * 2$



1.7 Exponents

Exponentiation is repeated multiplication. For example, $4^5 = 4*4*4*4*4 = 1024$. Exponentiation is easy to visualize when the exponent is 2 or 3.



$10^2 = 100$ $10^3 = 1000$

We can read 4^5 as "four to the power of five" or "four to the five". We often read 10^2 as "ten squared" and 10^3 as "ten cubed".

There are a few unintuitive rules for exponents. For any positive values of a, b, and c, the following will be true.

$$(a^b)^c = a^{b*c}$$
$$a^b * a^c = a^{b+c}$$
$$a^b * c^b = (a*c)^b$$
$$a^0 = 1$$
$$a^1 = a$$
$$0^a = 0$$
$$1^a = 1$$

I recommend checking your memory with simple examples. To check $(a^b)^c = a^{b*c}$, you could do the following:

$$
\begin{aligned}
(2^3)^4 &\stackrel{?}{=} 2^{3*4} \\
(2*2*2)^4 &\stackrel{?}{=} 2^{12} \\
(2*2*2)*(2*2*2)*(2*2*2)*(2*2*2) &\stackrel{?}{=} 2^{12} \\
2*2*2*2*2*2*2*2*2*2*2*2 &\stackrel{?}{=} 2^{12} \\
2^{12} &\stackrel{\checkmark}{=} 2^{12}
\end{aligned}
$$

1.8 Expanded-exponential form

Earlier, we saw that the positional notation could be expanded, like in the example below.

$$1234567 = \ 1*1000000 \ + \ 2*100000 \ + \ 3*10000 \ + \ 4*1000 \ + \ 5*100 \ + \ 6*10 \ + \ 7*1$$

With exponential notation, we can write this in expanded-exponential form.

$$1234567 = \ 1*10^6 \ + \ 2*10^5 \ + \ 3*10^4 \ + \ 4*10^3 \ + \ 5*10^2 \ + \ 6*10^1 \ + \ 7*10^0$$

1.9 Approximate multiplication

Rounded numbers are easier to multiply. Whenever you use a calculator to multiply two numbers, you should do a *sanity check* on the result. For example, imagine you are asked to multiply 7,302,842 and 491.

$$7302842 * 491 \approx \ ?$$

First, round each number to its largest non-zero place to get 7 million and 5 hundred.

$$(7000000) * (500)$$

Write each number in expanded notation.

$$(7 * 10^6) * (5 * 10^2)$$

Use the fact that when multiplying a bunch of numbers, the order does not matter.

$$7 * 5 * 10^6 * 10^2$$

It is easy to multiply 7 and 5. It is also easy to multiply the two powers of 10 by using the fact that $a^b * a^c = a^{b+c}$.

$$35 * 10^8$$

Thus, we expect the answer to be close to 3,500,000,000 (a few billion). The actual product is 3,585,695,422.

Practice approximate multiplication

1. Joseph claimed that 245^3 is 1,470,612. Determine 200^3 to show that he most likely messed up his calculation.

2. A field measures 934 feet by 429 feet. If each stalk of corn requires about 1 square foot, about how many stalks of corn are in the field? (hundreds? thousands? ten thousands? hundred thousands? millions?)

3. If you live 100 years, that is 36525 days. Each day has 86400 seconds. About how many seconds does an old person live? (millions? tens of millions? hundreds of millions? billions? tens of billions? hundreds of billions?...)

1.10 Going backwards

After adding numbers enough times, you might think to try the procedure in reverse. For example, you might ask, "What number plus 87 is 93?" This question could take the form of finding the value of x that made the following equation true.

$$x + 87 = 93$$

Solving the above equation is equivalent to determining what is left when 87 things are taken from 93 things.

$$x = 93 - 87$$

Again, the positional notation makes this arithmetic rather intuitive. We need to take away 8 tens and 7 ones. We start with 9 tens and 3 ones, but need to break a ten into ones, giving 8 tens and 13 ones. Thus, we are left with 6.

Similarly, you may ask, "What number times 7 is 56?"

$$x * 7 = 56$$

which would lead you to division.

$$x = 56 \div 7$$

Or, you may ask, "What number squared is 64?"

$$x^2 = 64$$

which would lead you to square roots.

$$x = \sqrt{64}$$

Messing with the exact numbers in the questions leads to negative, rational, and irrational (even imaginary) numbers.

Practice finding square roots of perfect squares

1. Evaluate $\sqrt{144}$

2. What number squared is 49?

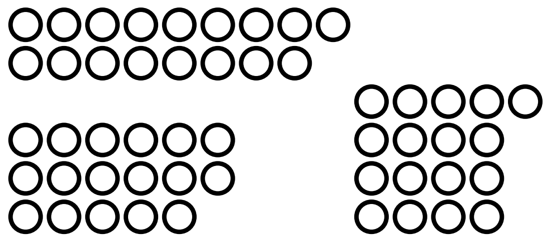
1.11 Factors, primes, and composites

When whole numbers are multiplied, we call them factors of the resulting product. Thus, because $2 * 3 * 10 = 60$ we know that 2, 3, and 10 are factors of 60. We also say that 60 is a multiple of 10. Other multiples of 10 are 0, 10, 20, 30, 40, 50...

A prime number only has itself and 1 as factors. Some examples of prime numbers follow:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29...$$

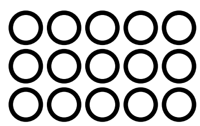
Thus, with 17 marbles, it is impossible to make a rectangular array.



A composite number has more than the two trivial factors. Some examples of composite numbers follow:

$$4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22...$$

Thus, it is possible to make a rectangular array with 15 marbles.



The number 1 is neither prime nor composite.

1.12 Prime factorization

Every positive integer can be uniquely expressed as a product of prime numbers. Some examples follow:

$$\begin{aligned} 60 &= 2^2 * 3 * 5 \\ 72 &= 2^3 * 3^2 \\ 273 &= 2 * 7 * 13 \\ 81 &= 3^4 \end{aligned}$$

Practice finding prime factorization

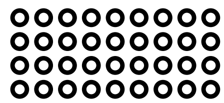
1. Express 126 as a product of primes.

2 Prime factorization, LCM, GCF

2.1 Factors and multiples

If three whole numbers are represented by x, y, and z such that $x * y = z$, then we say x and y are factors of z, z is a multiple of (or is divisible by) x, and z is a multiple of y.

For example, because $4 * 9 = 36$, we know the following: 4 and 9 are factors of 36; 36 is a multiple of 4; and 36 is a multiple of 9. From our understanding of multiplication, we also know 36 dots can be arranged in 4 rows and 9 columns.



The sequence of (positive) multiples of x is called "counting by x":

$$\text{multiples of } x = \{x,\ 2*x,\ 3*x,\ 4*x,\ 5*x,\ 6*x,\ 7*x...\}$$

The (positive) multiples of 5 are {5, 10, 15, 20, 25, 30...}. We call this sequence "counting by fives". It turns out that any whole number with a units digit of 0 or 5 is a multiple of 5. For example 3280 and 42835 are multiples of 5, and 5 is a factor of 7420.

There are a few divisibility rules that are worth memorizing.

Possible factor	A number is a multiple if:
2	The units digit is 2, 4, 6, 8, or 0
3	The sum of digits is a multiple of 3
5	The units digit is 0 or 5

For example, 276441 is a multiple of 3. We check this by adding the digits.

$$2 + 7 + 6 + 4 + 4 + 1 = 24$$

We either know 24 is a multiple of 3 or add its digits.

$$2 + 4 = 6$$

And we know 6 is a multiple of 3.

WARNING: this rule is special for 3 (and 9 has a similar rule). 7 does not have an easy rule.

Practice with divisibility rules

- Is 795 divisible by 3?

- Is 2345 divisible by 3?

2.2 Factor pairs

Some numbers can be factored in many different ways. For example, 144 has many pairs of factors:

$$
\begin{aligned}
144 &= 1 * 144 \\
144 &= 2 * 72 \\
144 &= 3 * 48 \\
144 &= 4 * 36 \\
144 &= 6 * 24 \\
144 &= 8 * 18 \\
144 &= 9 * 16 \\
144 &= 12 * 12
\end{aligned}
$$

Each of these pairs of factors suggests one way 144 marbles can be arranged into a rectangle. Notice that one of the two factors is always less than or equal $\sqrt{144}$. As the first factor increases, the second decreases, and they meet at 12.

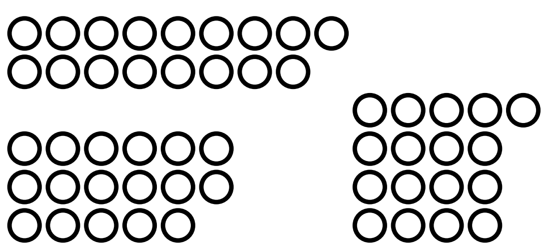
If we have three positive numbers represented by x, y, and z such that $x * y = z$ and $x < y$, then $x < \sqrt{z}$ and $y > \sqrt{z}$. Also, if $x * y = z$ and $x = y$, then $x = \sqrt{z}$ and $y = \sqrt{z}$.

Practice finding all factor pairs

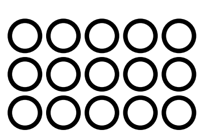
- Determine all factor pairs of 36.

2.3 Prime numbers and composite numbers

A prime number is only divisible by itself and 1. The first few primes are 2, 3, 5, 7, 11, 13, 17, 19, and 23. With a prime number of marbles, any attempt to make a rectangular array is futile. For example, 17 marbles do not arrange into rows and columns.



Numbers with more than 2 factors are called composites. The first few composites are 4, 6, 8, 9, 10, 12, 14, 15, and 16. Because 15 is composite, we can arrange 15 items into rows and columns.



The only even prime is 2, because all other even numbers are multiples of 2. 3 is the only prime that is a multiple of 3 (so 27 is not prime). 5 is the only prime that is a multiple of 5.

This reasoning leads to an ancient method for finding primes: the sieve of Eratosthenes.

In-class exercise: sieve of Eratosthenes

Cross out 1 (it is not prime). Circle 2 (it is prime). Cross out all multiples of 2. Circle 3 (it is prime). Cross out all multiples of 3.

Circle the next number that is not crossed out (it is prime). Cross out all multiples of that number. Repeat.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

You'll notice that every prime larger than 11 already has all its multiples crossed out. This is because $11^2 > 120$. Thus, any pair of factors of a number less than 11^2 has at least one that is less than 11. If this is confusing, refer back to factor pairs (section 2.2).

2.4 Primality checking

One method to determine whether a number is prime is to try dividing it by all primes less than its square root. For example, we could check whether 191 is prime.

- We first determine the nearest perfect squares to 191.

$$13^2 = 169$$

$$14^2 = 196$$

- Next, we determine whether 191 is divisible by 2, 3, 5, 7, 11, or 13.
  - 191 is not divisible by 2, as its unit digit is not 0, 2, 4, 6, or 8.
  - 191 is not divisible by 3, as its digits add to 11 - not a multiple of 3.
  - 191 is not divisible by 5, as its unit digit is not 0 or 5.
  - 191 is not divisible by 7, as 7*27 = 189.
  - 191 is not divisible by 11, as 11*17 = 187.
  - 191 is not divisible by 13, as 13*14 = 182.
- Thus, 191 is prime.

Practice checking for primality

1. Is 323 prime?

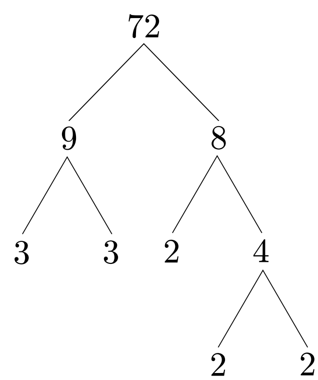
2. Is 397 prime?

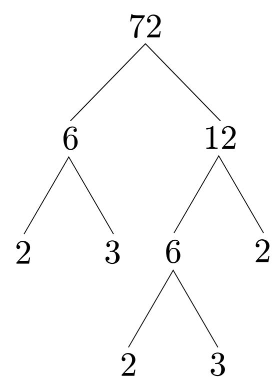
2.5 Prime factorization

Every composite number can be written as a product of primes. Some examples follow:

$$50 = 2 * 5^2$$
$$100 = 2^2 * 5^2$$
$$42 = 2 * 3 * 7$$
$$81 = 3^4$$

One method for determining the prime factorization is the *factor tree*. The goal is to break any composite number into factors until only primes remain. For example, a factor tree of 72 could look like either one below (or many other versions):





But, either way, the tree will show 72 is equal to $2^3 * 3^2$ (look at the ends of the branches).

Practice prime factorization

- Write 198 as a product of primes.

- Determine the prime factorization of 144.

- Determine the prime factorization of 221.

2.6 Factors with prime factorization

We have seen that every whole number can be represented in the prime-factorized form:

$$2^a * 3^b * 5^c * 7^d * 11^e * 13^f * 17^g * ...$$

where the exponents are whole numbers (whole means non-negative integers [0 is included]).

Given a specific whole number's prime factorization, we can express all factors of that number. A number is a factor if and only if its prime factorization has corresponding exponents that are less or equal.

For example, because we know that $72 = 2^3 * 3^2$, we can easily list all factors of 72:

$$2^0 * 3^0 \ (= 1)$$
$$2^1 * 3^0 \ (= 2)$$
$$2^2 * 3^0 \ (= 4)$$
$$2^3 * 3^0 \ (= 8)$$
$$2^0 * 3^1 \ (= 3)$$
$$2^1 * 3^1 \ (= 6)$$
$$2^2 * 3^1 \ (= 12)$$
$$2^3 * 3^1 \ (= 24)$$
$$2^0 * 3^2 \ (= 9)$$
$$2^1 * 3^2 \ (= 18)$$
$$2^2 * 3^2 \ (= 36)$$
$$2^3 * 3^2 \ (= 72)$$

Also, we can easily determine factor pairs. In the list above, the first goes with last, second with second-to-last, etc. For example, it is clear that $(2^2 * 3^1) * (2^1 * 3^1) = 2^3 * 3^2$. Remember that when multiplying powers with common bases, exponents add ($a^b * a^c = a^{b+c}$). Thus, factor pairs must have exponents that "add" up to the product's exponents.

Practice finding factor pairs in prime-factorized form

- Find 3 pairs of factors of the number represented by $2^4 * 5^7 * 13^{50}$. Write all numbers in their prime-factorized form.

2.7 Multiples with prime factorization

Let's start by considering the first few multiples of 72.

$$\begin{aligned}
1 * 72 &= 2^3 * 3^2 \\
2 * 72 &= 2^4 * 3^2 \\
3 * 72 &= 2^3 * 3^3 \\
4 * 72 &= 2^5 * 3^2 \\
5 * 72 &= 2^3 * 3^2 * 5 \\
6 * 72 &= 2^4 * 3^3 \\
7 * 72 &= 2^3 * 3^2 * 7 \\
8 * 72 &= 2^6 * 3^2 \\
9 * 72 &= 2^3 * 3^4 \\
10 * 72 &= 2^4 * 3^2 * 5
\end{aligned}$$

Notice that any multiple of 72 has 2 raised to a power greater than or equal to 3 and 3 raised to a power greater than or equal to 2. The discussion in section 2.6 should be sufficient to allow you to generalize in a formal way.

Practice stating a rule

- Write a rule that states what is true about any multiple of a number of the form:

$$2^a * 3^b * 5^c * 7^d * 11^e * 13^f * 17^g * ...$$

2.8 GCF - greatest common factor

To find the greatest/largest common/shared factor of two different numbers, we can list all factors of each, find the shared factors, and then pick the largest.

For example, we can find the greatest common factor of 175 and 245. We first determine their prime factorizations:

$$175 = 5^2 * 7$$

$$245 = 5 * 7^2$$

Thus we know their factors.

$$\text{Factors of } 175 \ = \{1, \ \ 5, \ \ 7, \ \ 5^2, \ \ 5*7, \ \ 5^2*7\}$$

$$\text{Factors of } 245 \ = \{1, \ \ 5, \ \ 7, \ \ 5*7, \ \ 7^2, \ \ 5*7^2\}$$

And we can see the largest factor is $5 * 7$, which equals 35.

Generally, if we have two whole numbers,

$$x = 2^a * 3^b * 5^c * 7^d * 11^e * ...$$

$$y = 2^A * 3^B * 5^C * 7^D * 11^E * ...$$

then we can find the greatest common factor:

$$\text{GCF}(x, y) = 2^{\min(a,A)} * 3^{\min(b,B)} * 5^{\min(c,C)} * 7^{\min(d,D)} * 11^{\min(e,E)} * ...$$

Practice finding GCF

1. Find the greatest common factor of 45 and 231.

2. Find the greatest common factor of 63 and 32.

2.9 LCM - least common multiple

The positive multiples of 5 are: {5, 10, 15, 20, 25, 30, 35, 40, 45, 50...}. The positive multiples of 6 are: {6, 12, 18, 24, 30, 36, 42...}. The lowest positive number that is a multiple of 5 and 6 is 30. We call 30 the least common multiple of 5 and 6. In this context, "least" means "lowest", and "common" means "shared" (some people misinterpret "least common" as "rare").

To find the least common multiple of 63 and 78 by listing multiples would be tedious. Instead, use prime factorization. We start by using factor trees to determine their prime factorizations.

$$63 = 3^2 * 7 \qquad\qquad 78 = 2 * 3 * 13$$

Every multiple of 63 will be a multiple of $3^2 * 7$, and so their prime factorization will contain a 3 to a power greater than or equal to 2 and a 7 to a power greater than or equal to 1.

$$\begin{aligned}
&63 * 1 = 3^2 * 7 && 78 * 1 = 2 * 3 * 13 \\
&62 * 2 = 2 * 3^2 * 7 && 78 * 2 = 2^2 * 3 * 13 \\
&63 * 3 = 3^3 * 7 && 78 * 3 = 2 * 3^2 * 13 \\
&63 * 4 = 2^2 * 3^2 * 7 && 78 * 4 = 2^3 * 3 * 13 \\
&63 * 5 = 3^2 * 5 * 7 && 78 * 5 = 2 * 3 * 5 * 13 \\
&63 * 6 = 2 * 3^3 * 7 && 78 * 6 = 2^2 * 3^2 * 13 \\
&63 * 7 = 3^2 * 7^2 && 78 * 7 = 2 * 3 * 7 * 13 \\
&\quad\vdots && \quad\vdots
\end{aligned}$$

It should not seem too surprising that the least common multiple is $2 * 3^2 * 7 * 13$. We want each prime to have as small an exponent as possible while keeping the product a multiple both 63 and 78.

Generally, if we have two whole numbers,

$$x = 2^a * 3^b * 5^c * 7^d * 11^e * ...$$

$$y = 2^A * 3^B * 5^C * 7^D * 11^E * ...$$

then we can find the least common multiple:

$$\text{LCM}(x, y) = 2^{\max(a,A)} * 3^{\max(b,B)} * 5^{\max(c,C)} * 7^{\max(d,D)} * 11^{\max(e,E)} * ...$$

Practice finding LCM

1. Find the least common multiple of 45 and 231.

2. Find the least common multiple of 63 and 32.

Homework!

1. Find LCM and GCF of 100 and 330.

2. Find LCM and GCF of 48 and 252.

3. Find LCM and GCF of 91 and 130.

4. Find LCM and GCF of 323 and 437.

5. Find LCM and GCF of 80 and 1024.

6. Find LCM and GCF of 378 and 405.

7. Find LCM and GCF of $13^{97} * 17^{30} * 23^{88}$ and $3^{3} * 13^{14} * 17^{77}$.

3 Division, remainders, and fractions

3.1 Division

The physical interpretation of division is splitting items into piles. For example, if we split 56 pebbles into 7 piles, each pile would contain 8 pebbles. Thus, we say $56 \div 7 = 8$. We could also imagine grouping 56 pebbles into 7 rows and 8 columns.

Division is the reverse of multiplication. Thus, the two equations below would have the same solution.

$$7 * x = 56$$

$$x = 56 \div 7$$

We often see two other ways of writing division.

$$56 \div 7 = 56/7 = \frac{56}{7}$$

In division, order matters. Thus, in the equation $x \div y = z$, we give each number a unique name. We call x the dividend, y the divisor, and z the quotient.

Practice simple division (dividend, divisor, and quotient are all whole)

1. Jennifer has 66 gummy bears to share between herself and five of her friends. How many gummy bears does each person get?

2. Kirill has 20 boxes of apples, and each box has 8 apples. If he splits the apples between his 10 horses, how many apples does each horse eat?

3.2 Remainders

Division runs into trouble. For example, when 23 candies are split between 6 people, each person can get 3 (using up 18), but there are 5 candies remaining.

Determining the remainder with a calculator is harder than you might guess. One procedure is outlined below with the example of finding the remainder when 23 is divided by 6.

- Enter 23/6 into calculator. The calculator will provide a decimal approximation.

$$20/6 = 3.833333333$$

- Truncate the result (round the result down to an integer) without a calculator. We often call the function that rounds down "floor".

$$\text{floor}(3.833333333) = 3$$

- Determine the product of the divisor (6) and the floored result (3).

$$6 * 3 = 18$$

- Take a difference between the dividend (23) and the resulting product (18).

$$23 - 18 = 5$$

Practice finding remainders

1. If 9 boxes each have 12 cans, and the cans are split between 25 people (with the remainder set aside), how many cans does each person get?

2. If 9 boxes each have 12 cans, and the cans are split between 25 people (with the remainder set aside), how many cans are remaining?

3.3 Non-whole quantities

Often, the items we are dividing can be cut into evenly sized pieces. For example, 4 people can share 3 pizzas by cutting the pizzas into slices. Everyone eats a bit less than a whole pizza. In fact, if we cut each pizza into 4 slices, each person can eat 3 slices. We say each person eats three fourths of a pizza. We now focus on these fractional quantities.

3.4 Terminology and notation

A **fraction** has the form:

$$\frac{a}{b}$$

where a and b are both whole numbers. We call a the **numerator** and b the **denominator**. There are two types of fractions, proper fractions and improper fractions.

In **proper fractions** $a < b$ (the denominator is larger). An example of a proper fraction is three eighths:

$$\frac{3}{8}$$

In **improper fractions** $a > b$ (the numerator is larger). An example of a improper fraction is eleven eighths:

$$\frac{11}{8}$$

A **mixed number** is the sum of a whole number (W) and a proper fraction ($\frac{N}{D}$).

$$W + \frac{N}{D}$$

An example of a mixed number is two and three fourths.

$$2 + \frac{3}{4}$$

Often, mixed numbers are written with an implicit "+".

$$2\frac{3}{4}$$

Beware: two adjacent expressions are usually assumed to be multiplied, not added.

3.5 Some trivial equivalencies

Some equivalencies are fundamental. The following are true for any value of x and any non-zero value of y.

$$x = \frac{x}{1}$$

$$1 = \frac{y}{y}$$

This second equation suggests there are infinite ways of writing the number one. For example, 247/247 is equivalent to 1.

3.6 Multiplying fractions

When two fractions are multiplied, the following rule is followed:

$$\frac{a}{b} * \frac{c}{d} = \frac{a * c}{b * d}$$

For example, we can determine that two sevenths times five thirds is ten twenty-oneths:

$$\frac{2}{7} * \frac{5}{3} = \frac{2 * 5}{7 * 3} = \frac{10}{21}$$

3.7 Equivalent fractions

Because there are infinite fractional ways of writing the number 1 (see section 3.5), the rule for multiplication suggests that any fractional quantity can be expressed multiple ways. For any values of a, b, and y:

$$\frac{a}{b} = \frac{a * y}{b * y} \tag{3.7.1}$$

We can write more intermediary equalities to see why.

$$\frac{a}{b} = \frac{a}{b} * 1 = \frac{a}{b} * \frac{y}{y} = \frac{a * y}{b * y}$$

For example,

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \ldots$$

This also means any integer can be written in infinite ways.

$$x = \frac{x}{1} = \frac{x * y}{y}$$

For example,

$$5 = \frac{5}{1} = \frac{10}{2} = \frac{15}{3} = \frac{20}{4} = \ldots$$

3.8 Simplifying fractions

Let's consider a common fraction:

$$\frac{a}{b}$$

If the numerator (a) and denominator (b) have a greatest common factor of 1, we say the fraction is **simplified**. Also, if the fraction can be expressed as an integer, we prefer to do that.

When the greatest common factor is bigger than 1, we simplify the fraction by factoring out the greatest common factor from the numerator and denominator, and using the equivalency shown in equation 3.7.1 "backwards".

For example, we can simplify 30/24.

$$\frac{30}{24}$$

$$\text{GCF}(30, 24) = 6$$

$$\frac{30}{24} = \frac{5 * 6}{4 * 6} = \frac{5}{4}$$

Practice simplifying fractions

- Write $\frac{108}{72}$ as a simplified fraction.

- Write $\frac{196}{210}$ as a simplified fraction.

- Write $\frac{12}{35} * \frac{25}{18} * \frac{21}{55}$ as a simplified fraction.

3.9 Simplifying products of fractions

When given a product of fractions, it can be beneficial to use prime factorization. For example,

$$\frac{12}{35} * \frac{25}{18} * \frac{21}{55}$$

$$\frac{2^2 * 3}{5 * 7} * \frac{5^2}{2 * 3^2} * \frac{3 * 7}{5 * 11}$$

$$\frac{2^2 * 3^2 * 5^2 * 7}{2 * 3^2 * 5^2 * 7 * 11}$$

$$\frac{2}{11} * \frac{2 * 3^2 * 5^2 * 7}{2 * 3^2 * 5^2 * 7}$$

$$\frac{2}{11}$$

3.10 Division of fractions

When fractions are divided, we can rewrite the expression as the dividend multiplied by the reciprocal of the divisor.

$$\frac{a}{b} \div \frac{c}{d} \; = \; \frac{a}{b} * \frac{d}{c}$$

Thus, the following equality is always true.

$$\frac{a}{b} \div \frac{c}{d} \; = \; \frac{a * d}{b * c}$$

We can also write division of fractions as fractions of fractions.

$$\frac{\frac{a}{b}}{\frac{c}{d}} \; = \; \frac{a}{b} * \frac{d}{c} \; = \; \frac{a * d}{b * c}$$

Practice dividing fractions

- Divide 6/35 by 21/10. Express your answer as a simplified fraction.

3.11 Addition and subtraction of fractions

Adding or subtracting fractions requires a common denominator. For example, it is clear that

$$\frac{2}{12} + \frac{9}{12} = \frac{11}{12}$$

However, determining the sum is less clear if the addends were given in simplified form.

$$\frac{1}{6} + \frac{3}{4} = \ ?$$

A brute strategy could be followed.

$$\frac{a}{b} + \frac{c}{d} \quad = \quad \frac{a}{b} * \frac{d}{d} + \frac{c}{d} * \frac{b}{b} \quad = \quad \frac{ad}{bd} + \frac{bc}{bd} \quad = \quad \frac{ad + bc}{bd}$$

In the example above, this would look like:

$$\frac{1}{6} + \frac{3}{4}$$

$$\frac{4}{24} + \frac{18}{24}$$

$$\frac{22}{24}$$

$$\frac{11}{12}$$

But, notice we ended up with larger numbers than if we used 12 as the denominator, and we had to simplify the answer.

The elegant strategy is to find the least common multiple of the original denominators. We call this the **least common denominator**.

$$\frac{1}{6} + \frac{3}{4}$$

$$\frac{2}{12} + \frac{9}{12}$$

$$\frac{11}{12}$$

Again, prime factorization can be useful.

$$\frac{3}{175} - \frac{2}{245}$$

$$\frac{3}{5^2 * 7} - \frac{2}{5 * 7^2}$$

$$\frac{3 * 7}{5^2 * 7^2} - \frac{2 * 5}{5^2 * 7^2}$$

$$\frac{11}{1225}$$

3.12 Practice adding fractions

- Express $\frac{5}{6} + \frac{2}{9}$ as a simplified fraction.

- Express $\frac{5}{72} + \frac{7}{108}$ as a proper fraction.

- Express $\frac{42}{5} - \frac{14}{2}$ as a simplified fraction.

- Express $5 + \frac{2}{7}$ as an improper fraction.

3.13 Converting mixed numbers into improper fractions

The rule for addition allows us to convert mixed numbers into improper fractions.

$$x + \frac{y}{z} = \frac{xz + y}{z}$$

This equivalence can be shown in more steps.

$$x + \frac{y}{z} \quad = \quad \frac{x}{1} + \frac{y}{z} \quad = \quad \frac{xz}{z} + \frac{y}{z} \quad = \quad \frac{xz + y}{z}$$

So, for example:

$$2\frac{3}{4}$$

$$2 + \frac{3}{4}$$

$$\frac{8}{4} + \frac{3}{4}$$

$$\frac{11}{4}$$

After this is done enough times, you can probably make a bigger jump.

$$5\frac{1}{6}$$

$$\frac{5 * 6 + 1}{6}$$

$$\frac{31}{6}$$

Practice converting mixed into improper

- Write $8\frac{1}{3}$ as an improper fraction.

- Write $11\frac{4}{7}$ as an improper fraction.

3.14 Converting improper to mixed

To convert improper fractions to mixed numbers, we follow the procedure for finding remainder. The remainder becomes the numerator of the proper fraction of the mixed number. The denominator of the improper fraction becomes the denominator of the proper fraction of the mixed number. The floored division becomes the integer of the mixed number.

For example:

$$\frac{74}{13}$$

$$\text{floor}\left(\frac{74}{13}\right) = 5$$

$$74 - 5 * 13 = 9$$

$$5 + \frac{9}{13}$$

Thus, we see that 74/13 can be expressed as $5\frac{9}{13}$.

Practice converting improper fractions into mixed numbers

- Write 17/3 as a mixed number.

- Write 25/4 as a mixed number.

- Write 64/5 as a mixed number.

HOMEWORK!

1. A deck of cards has 54 cards, but two of them are jokers. The non-jokers are evenly split into four suits. How many cards are in each suit?

2. Lana bought 8 packs of shirts. Each pack has 3 shirts. She wants herself and each of her four friends to have the same number. How many shirts does each person get? How many shirts are left over?

3. A recipe for a cake requires $2\frac{1}{3}$ cups sugar. How many cups of sugar are needed to make 4 cakes? Express your answer as a mixed number.

4. Let's assume 2/3 of people like tacos, and 5/7 of them like cheese on their tacos. That means that $\frac{2}{3} * \frac{5}{7}$ of people like cheesy tacos. Express the fraction of people that like cheesy tacos as a proper fraction.

5. Mallory has a string that is $3\frac{2}{3}$ meters long. Nate has a string that is $5\frac{4}{7}$ meters long. What is the total length of string, expressed as a mixed number?

6. A recipe calls for $3\frac{1}{2}$ cups of sugar for a dozen muffins. How much sugar is in each muffin? Express your answer as a fraction.

4 Arithmetic with mixed numbers

4.1 Adding mixed numbers

A mixed number is a sum of a whole number and a proper fraction. Below, the whole number is 3 and the fraction is $\frac{4}{5}$.

$$3\frac{4}{5} = 3 + \frac{4}{5}$$

We add two mixed number by adding the wholes, adding the fractions, and possibly carrying if the resulting fraction is larger than 1.

Examples:

- Add $3\frac{4}{5}$ and $7\frac{2}{5}$

  $$3 + \frac{4}{5} + 7 + \frac{2}{5}$$

  Commute and associate terms.

  $$(3 + 7) + \left(\frac{4}{5} + \frac{2}{5}\right)$$

  Evaluate expressions in parentheses.

  $$(10) + \left(\frac{6}{5}\right)$$

  Because $\frac{6}{5} > 1$, rewrite the fraction as a mixed number (carry).

  $$(10) + \left(1 + \frac{1}{5}\right)$$

  Now, remove parentheses and add the whole numbers.

  $$11 + \frac{1}{5}$$

  If you want, you can write the mixed number without the "+".

  $$11\frac{1}{5}$$

- Add $4\frac{2}{3}$ and $6\frac{1}{4}$.

  $$(4 + 6) + \left(\frac{2}{3} + \frac{1}{4}\right)$$

  Use a common denominator.

  $$(4 + 6) + \left(\frac{8}{12} + \frac{3}{12}\right)$$

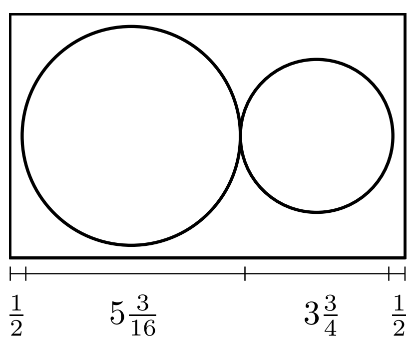
  $$10 + \frac{11}{12}$$

  The answer can also be written as $10\frac{11}{12}$.

Practice adding mixed numbers

- A recipe for cake requires $2\frac{1}{2}$ cups of flour, and a recipe for muffins requires $3\frac{2}{3}$ cups of flour. How many cups of flour are needed to follow each recipe once?

- A rectangular slot is to be cut in a wall to allow two adjacent pipes to run between rooms. The pipes have diameters of $5\frac{3}{16}$ and $3\frac{3}{4}$ inches. To make a half-inch wiggle room on both sides, how wide should the rectangular slot be?



4.2 Subtracting mixed numbers

To subtract mixed numbers, we can subtract the wholes and fractions separately. Subtracting mixed numbers sometimes requires "borrowing".

Example:

- Subtract $4\frac{1}{2}$ from $7\frac{1}{12}$.

$$\left(7+\frac{1}{12}\right)-\left(4+\frac{1}{2}\right)$$

Find a common denominator for the fractions.

$$\left(7+\frac{1}{12}\right)-\left(4+\frac{\mathbf{6}}{\mathbf{12}}\right)$$

Because $\frac{1}{12}<\frac{6}{12}$, borrow from the 7.

$$\left(6+1+\frac{1}{12}\right)-\left(4+\frac{6}{12}\right)$$

$$\left(6+\frac{12}{12}+\frac{1}{12}\right)-\left(4+\frac{6}{12}\right)$$

$$\left(6+\frac{13}{12}\right)-\left(4+\frac{6}{12}\right)$$

Subtract wholes and fractions.

$$2+\frac{7}{12}$$

The answer can also be written as $2\frac{7}{12}$.

Practice subtracting mixed numbers

- Jeremy has $6\frac{1}{3}$ cups of flour. He uses $1\frac{1}{2}$ cups of flour for cookies. How much flour remains?

Multiplying mixed numbers

Typically, to multiply mixed numbers, people convert to improper fractions, multiply the improper fractions, and convert back to mixed numbers. Below we see how we can multiply mixed numbers more directly. This technique uses the fact that a product of sums can be expanded into a sum of products (FOIL = Firsts, Outsides, Insides, Lasts).

$$(a + b) * (c + d) = ac + ad + bc + bd$$

So, for example, we may want to multiply $3\frac{2}{7}$ and $5\frac{4}{5}$.

$$\left(3 + \frac{2}{7}\right) * \left(5 + \frac{4}{5}\right)$$

Exand (FOIL).

$$15 \ + \ \frac{12}{5} \ + \ \frac{10}{7} \ + \ \frac{8}{35}$$

Find common denominator.

$$15 \ + \ \frac{84}{35} \ + \ \frac{50}{35} \ + \ \frac{8}{35}$$

Sum the fractions.

$$15 \ + \ \frac{142}{35}$$

Convert improper fraction to mixed number.

$$15 \ + \ 4 \ + \ \frac{2}{35}$$

Thus, our answer is $19\frac{2}{35}$.

Practice multiplying mixed numbers

- A standard sheet of paper in most the world (not the U.S.) has dimensions of about $8\frac{1}{4}$ and $11\frac{2}{3}$ inches. Using those values, calculate the area in square inches.

4.3 Multiplying mixed number with whole number

A whole number is a special case of a mixed number, where the fraction is equal to zero. Thus, we can use the strategy for multiplying mixed numbers to multiply a mixed number by a whole number.

For example, if we want to multiply 5 and $3\frac{2}{7}$:

$$(5) * \left(3 + \frac{2}{7}\right)$$

$$15 + \frac{10}{7}$$

$$16 + \frac{3}{7}$$

Practice multiplying mixed number with whole number

1. Raul has 20 jars, each with capacity of $2\frac{3}{4}$ liters. What is the total capacity of all the jars together?

2. Each change of 1°C is $1\frac{4}{5}$°F. If the temperature drops by 9°C, how many degrees Fahrenheit does the temperature decrease?

4.4 Dividing mixed numbers

When dividing mixed numbers, we rewrite each mixed number as an improper fraction and then multiply the dividend by the recipricol of the divisor.

For example, we might want to evaluate $\left(5+\frac{2}{3}\right) \div \left(2+\frac{4}{7}\right)$ as a mixed number.

$$\frac{5+\frac{2}{3}}{2+\frac{4}{7}}$$

We convert mixed numbers to improper fractions.

$$\frac{\frac{17}{3}}{\frac{18}{7}}$$

We use the fact that $\frac{a}{b} = a * \frac{1}{b}$, division is equivalent to multiplying by the recipricol of the divisor.

$$\frac{17}{3} * \frac{7}{18}$$

$$\frac{119}{54}$$

$$2 + \frac{11}{54}$$

which you can write as $2\frac{11}{54}$.

Practice dividing mixed numbers

- If Kinomi has $42\frac{3}{4}$ yards of rope, and the rope weighs $13\frac{1}{4}$ pounds, what is the linear density of the rope in pounds per yard expressed as a mixed number?

- Jenny is carrying water from a river to her house. If Jenny can carry $3\frac{5}{8}$ gallons, but needs to collect $30\frac{1}{8}$ gallons, how many trips does Jenny need to make?

4.5 Dividing mixed number by whole number

Dividing a mixed number by a whole number can be simpler, because $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$. For example, if we divide $21\frac{2}{3}$ by 5, and want the result as a mixed number:

$$\frac{21 + \frac{2}{3}}{5}$$

Use the identity $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

$$\frac{21}{5} + \frac{\frac{2}{3}}{5}$$

Division can be rewritten as multiplication by reciprocal of divisor.

$$\frac{21}{5} + \frac{2}{3} * \frac{1}{5}$$

$$\frac{21}{5} + \frac{2}{15}$$

Determine a common denominator.

$$\frac{63}{15} + \frac{2}{15}$$

$$\frac{65}{15}$$

Convert to mixed number.

$$4 + \frac{5}{15}$$

$$4 + \frac{1}{3}$$

You can write the answer as $4\frac{1}{3}$.

Practice dividing mixed number by whole number

- When a $25\frac{1}{2}$ meter plank is sawed into 8 equally sized pieces, how long is each piece?

4.6 Dividing a whole number by a mixed number

It must be emphasized that division does **not** distribute over a sum in the divisor.

$$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$$

So, a mixed number in the denominator is first converted to an improper fraction. For example, if we divide 15 by $1\frac{1}{3}$, and want to express the result as a mixed number:

$$\frac{15}{1+\frac{1}{3}}$$

Convert the denominator to an improper fraction.

$$\frac{15}{\frac{4}{3}}$$

Division can be rewritten as multiplication of dividend by reciprocal of divisor.

$$15 * \frac{3}{4}$$

$$\frac{15}{1} * \frac{3}{4}$$

$$\frac{45}{4}$$

Convert improper fraction to mixed number.

$$11 + \frac{1}{4}$$

The answer can also be written as $11\frac{1}{4}$.

Practice dividing whole number by mixed number

At 4°C, water has a density of $1\frac{47}{50}\ \frac{\text{slugs}}{\text{ft}^3}$. How many cubic feet of water is equivalent to 97 slugs of water?

5 Proportionality, rates, and conversion factors

If you walk twice the time, you walk twice as far. If you buy triple the bananas, you pay triple the cost. If you quadrupel the peanuts, you quadrupel the weight.

These situations display proportionality. During changes in amount, something has remained constant in each case. I assumed that velocity of walking, per-unit cost of bananas, and density of peanuts did not change. We call these rates.

5.1 Rates

It can be useful to think about units. Rates usually have units of one unit divided by another. In the walking example, the three useful quantitative measures are time, distance, and velocity. The units for time could be seconds, the units for distance could be meters, and the units for velocity could be meters per second. These units correctly suggest we can determine velocity by dividing distance by time.

$$v = d/t$$

Notice the units work out, such that both sides of the equation have units of $\frac{\mathrm{m}}{\mathrm{s}}$. Also, notice that variables are typed as italicized text, whereas abbreviations of units use upright, roman text.

$$\left(\frac{\mathrm{m}}{\mathrm{s}}\right) = (\mathrm{m})/(\mathrm{s})$$

We can write other equations that capture this relationship.

$$d = vt$$

If we do some unit analysis, we see both sides have units of meters.

$$(\mathrm{m}) = \left(\frac{\mathrm{m}}{\mathrm{s}}\right) * (\mathrm{s})$$

Also,

$$t = \frac{d}{v}$$

Now, both sides have units of seconds.

$$\mathrm{s} = \frac{\mathrm{m}}{\frac{\mathrm{m}}{\mathrm{s}}} \qquad \left(= \mathrm{m} * \frac{\mathrm{s}}{\mathrm{m}}\right)$$

Practice!

- Write three equations that relate total cost, number of bananas, and per-banana price. Also, show that units match on either side of each equation.

5.2 Conversion factors

There are various units of length: meters, inches, feet, yards, kilometers, centimeters, miles, light years, etc. There are various units of time: seconds, minutes, hours, days, weeks, months, years, centuries, millenia, etc.

We all know that one week is equivalent to seven days. This equivalence can be used as a rate of conversion $\left(\frac{7\text{ days}}{1\text{ week}}\right)$. So, if we wanted to convert 6 weeks into days:

$$(6\text{ weeks}) * \left(\frac{7\text{ days}}{1\text{ week}}\right) = 42\text{ days}$$

We can also use the reciprocal of the conversion factor to convert days to weeks. For example, if we wanted to know how many weeks are equivalent to 84 days.

$$(84\text{ days}) * \left(\frac{1\text{ week}}{7\text{ days}}\right) = 12\text{ weeks}$$

These conversion factors can be strung together into a single expression. For example, if we wanted to convert 15 $\frac{\text{feet}}{\text{hour}}$ into $\frac{\text{inches}}{\text{second}}$, we could evaluate the following.

$$\left(\frac{15\text{ ft}}{\text{hr}}\right)\left(\frac{12\text{ in}}{1\text{ ft}}\right)\left(\frac{1\text{ hr}}{60\text{ min}}\right)\left(\frac{1\text{ min}}{60\text{ sec}}\right) = 0.05\ \frac{\text{in}}{\text{sec}}$$

Practice

Use the following (sometimes approximate) equivalences:

$$\begin{aligned} 2.54\text{ cm} &= 1\text{ in} \\ 10,000\ \mu\text{m} &= 1\text{ cm} \\ 1461\text{ days} &= 4\text{ years} \end{aligned}$$

- The moon's average distance is growing by 1.5 inches per year. Find this speed in $\frac{\mu\text{m}}{\text{day}}$.

6 (Hard) practice midterm problems

1. A rectangular flower bed has a width of 3 **feet** and a length of 5 **feet**. If each seed needs 9 **square inches** of area, how many seeds should be planted in the bed?

2. Find the perimeter of a rectangle whose width is 4 meters and whose area is 24 square meters.

3. When $3\frac{1}{2}$ cups flour are used for 12 muffins, how many cups are needed for 27 muffins (answer as mixed number)?

4. If $23\frac{1}{3}$ feet of string is cut into $1\frac{1}{6}$ feet pieces, how many pieces will there be?

5. Last year I read $2\frac{3}{4}$ articles each day (on average). This year I read $5\frac{1}{2}$ articles each day. Comparing this year to last, how many more articles do I read each month (assume 1 month = 30 days).

6. As Germa races a go-kart, she completes 3 laps every 10 minutes. She bought 120 minutes worth of racing, but only completed 15 laps. How many more minutes does Germa have?

7. Ferndo bought 8 dozen bagels for his office. 78 bagels were eaten. How many dozen bagels are remaining? (answer as a mixed number)

8. If I buy 4 comics every 3 days, then how many days does it take for me to buy 68 comics?

9. I started with $35. I bought 8 carrots for $0.20 each, a dozen eggs for $3.78, and 5 boxes of cereal for $4.67 each. Then, I found 2 pennies. How much money do I have?

10. A house has a lawn that is 120 feet by 50 feet. If it takes 5 minutes to mow 350 square feet, what fraction of an hour will the lawn take to mow (answer as a fraction)?

11. If you sleep 8 hours each day, and live 85 years, how many years would you have slept (answer as a mixed number)?

12. If candies cost 22 cents each, how many can I buy with 12 dollars?

13. With an income of \$1800 per month, and rent using 1/3 of income, and food using 1/2 of income, how much money can be spent each year on other things than income and food?

14. If my dog eats $1\frac{5}{8}$ pounds of food every day, how many days will a 26 pound bag last?

15. I want to eat 1/4 of a pizza, and George wants to eat 1/3 of a pizza. Assuming we chop the pizza into equally sized slices, how many cuts (across pizza, through middle) should we make?

16. I used to drink $3\frac{1}{2}$ sodas each day. Now I drink $1\frac{1}{4}$ sodas each day. If I can buy sodas for \$5.89 for a pack of 10, then how much money am I saving each week by drinking less soda?

17. A car travels at 55 miles per hour. How long will it take the car to go 200 miles?

18. A square has sides of length 6 feet. Find the area and perimeter of the square.