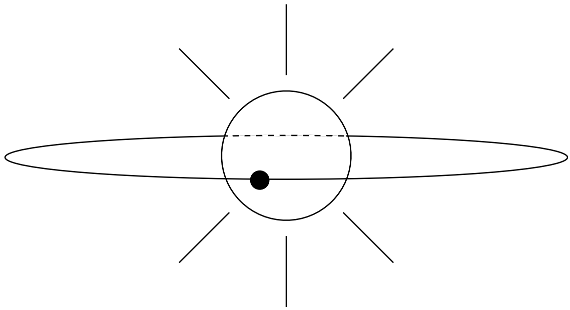
Exoplanets

Transit photometry is by far the most successful method for detecting exoplanets, planets outside our solar system. Basically, if the exoplanet gets between its star and us, that star will dim slightly.



We can predict the amount of dimming with simple geometry. From such a far distance, the star and exoplanet will appear as circles with relative radii equal to their true relative radii (there are no perspective effects). If we assume that all parts of the star's circle are equally luminous, then we just need to know what fraction of the star's circle is covered by the exoplanet's circle. We will use $A_\bullet$ as the exoplanet's circular (or cross-sectional) area, $A_☼$ as the star's circular area, $r_\bullet$ as the radius of the exoplanet, and $r_☼$ as the radius of the star. By remembering the formula for the area of a circle ($A = \pi r^2$), we can see that the fraction of the light that is blocked can be determined by taking the ratio of the squares of the radii.

$$\frac{A_\bullet}{A_☼} = \frac{r_\bullet^2}{r_☼^2}$$

For an earth-sized exoplanet around a sun-sized star, $r_\bullet \approx 6 \times 10^3$ km and $r_☼ \approx 7 \times 10^5$ km. With those specific numbers we find:

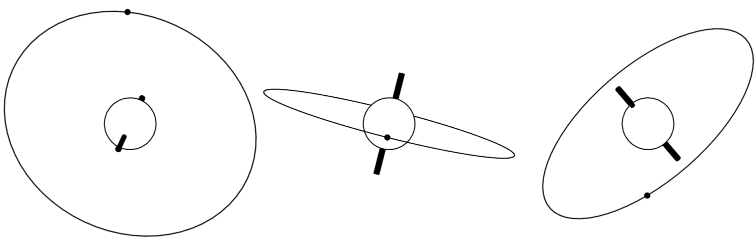
$$\frac{A_\bullet}{A_☼} = \frac{(6 \times 10^3)^2}{(7 \times 10^5)^2} = \frac{36 \times 10^6}{49 \times 10^{10}} \approx 1 \times 10^{-4} = \frac{1}{10,000}$$

Thus, we estimate that to detect an earth-sized exoplanet around a sun-sized star, our instruments need to be accurate enough to detect a 1 out of 10,000 change in brightness. Fortunately, telescopes like Kepler are capable of such feats.

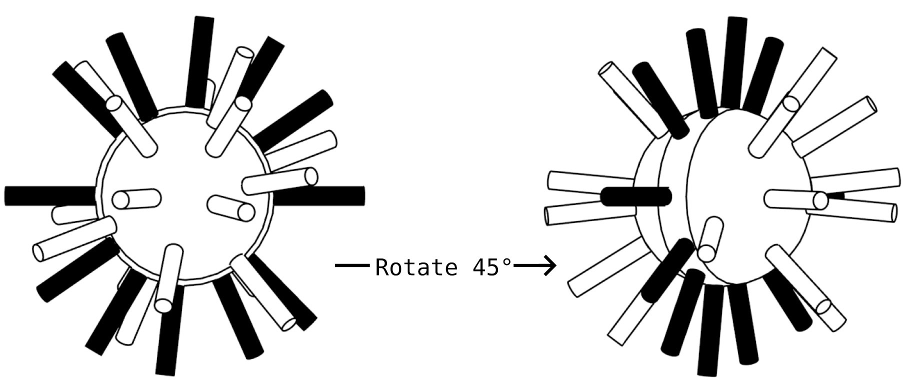
Chance to See Transit

Now for something a bit more challenging. Let's imagine an exoplanet orbits a star, but the plane of the orbit is randomly chosen. What is the probability, P, that the exoplanet will orbit in such a way as to periodically dim the star (from our point of view)?

We will assume circular orbits. It is easier to picture all the possible planes of orbit by focusing on the possible axes of rotation (the thick bars are the axes).



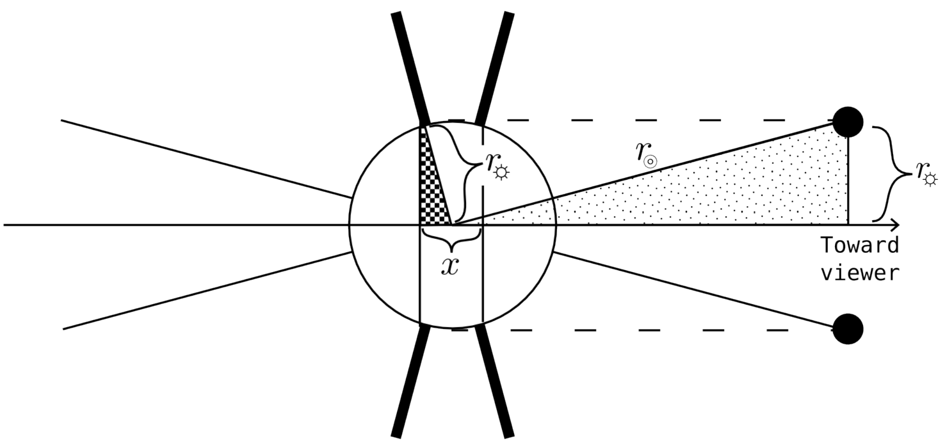
Picking a random plane corresponds to picking a random axis. Of the three images shown above, only the middle one shows an orbit that will be detectable via transit photometry. If we picture rotating the axis around in every possible orientation, we realize only a small fraction will put the orbit in front of the star. In fact, each of the compatible orbits has an axis extending through a circular strip (zone). Below, the black cylinders represent compatible orbits while the white cylinders represent incompatible orbits.



So, our problem is equivalent to finding the fraction of the sphere's area covered by the strip. Because we are now considering surface areas, and not cross-sectional areas, we will use new symbols. The surface area of the star is $S_{\sun}$ and the surface area of the strip is $S_{||}$.

$$P = \frac{S_{||}}{S_{\sun}}$$

But we first need to determine how wide the strip is. The strip's width depends on how far an orbit can tilt while remaining between the viewer and the star. We will call an orbit that is tilted as much as possible while still being detectable a critically tilted orbit. The width of the strip, x, depends on the radius of the orbit, $r_{\circledcirc}$, and the radius of the star, $r_{\sun}$. If we imagine everything (star, viewer, and critically tilted orbits) from the side (rotated a full 90° with axes in the plane of the paper), we see two similar triangles (checkered and polkadotted).



Notice the critically tilted orbit raises the exoplanet to the star's radius above the line connecting the center of the star and the viewer. Corresponding pairs of sides of similar triangles have equivalent ratios of lengths, so:

$$x = \frac{2r_{\sun}^2}{r_{\circledcirc}}$$

We are one fun fact away from the answer. If you had a perfectly spherical potato and sliced it into equally thick slabs, every slab would have the same amount of skin. This means that every time we add some amount to x, $S_{||}$ grows by a consistent amount, regardless of how big x already is. Proving this fun fact is a bit tedious (see the following section), but if we just accept the fun fact about spheres, the area of the strip, $S_{||}$, is therefore the surface area of the star multiplied by the ratio of x to the diameter.

$$S_{||} = S_{\star} \frac{x}{2r_{\star}}$$

And so, by combining and rearranging the previous three equations,

$$P = \frac{r_{\star}}{r_{\odot}}$$

For an earth-like orbit around a sun-like star, $r_{\odot} = 1.5 \times 10^8$ km and $r_{\star} = 7 \times 10^5$ km. Thus, the chance that a planet in an earth-like orbit would dim a sun-like star from our point of view is:

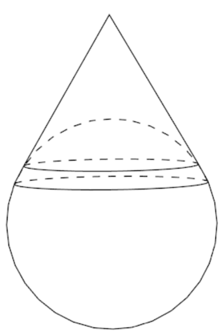
$$P = \frac{7 \times 10^5}{1.5 \times 10^8} \approx 5 \times 10^{-3}$$

or, about half a percent chance. Luckily there are lots of stars! This means that even though we haven't detected a planet around a given star, a planet could very well be there.

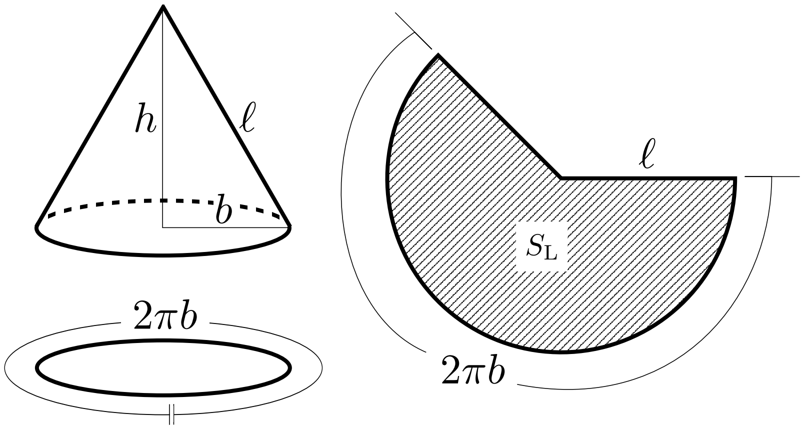
Proof of Potato Principle

If we already knew the general equation for the differential area of a surface of revolution, this would actually be quite easy. If you know it, you can skip to the last couple lines.

If we slice a spherical potato into very thin sections (like for making potato chips), the shape of each slice could be estimated very closely as a section of a cone. The differential element is therefore a section (frustum) of a cone. Cones are a bit straighter than spheres, so dealing with them is a bit easier.



We want an equation for the lateral surface area of a cone (total surface minus circular surface). The lateral surface could be cut along a lateral edge, ℓ, and flattened into a circular sector with radius ℓ and an arc length equivalent to the basal circle's circumference, $2\pi b$.

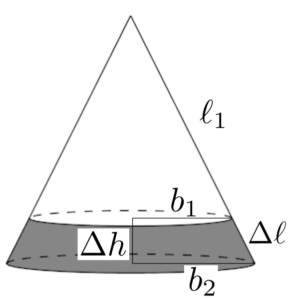


So, the area of that circular sector, S_{L} (which is equivalent to the lateral area of the cone), can be determined as a fraction of a whole circle.

$$S_{\mathrm{L}} = \pi\ell^2\left(\frac{2\pi b}{2\pi\ell}\right)$$

$$S_{\mathrm{L}} = \pi\ell b$$

Now, if we want the frustum's lateral area, S_{F}, we take a difference between two overlapping cones' lateral areas.



$$S_{\mathrm{F}} = \pi b_2\left(\ell_1 + \Delta\ell\right) - \pi b_1\ell_1$$

Similar triangles tell us the following.

$$\frac{\ell_1 + \Delta\ell}{b_2} = \frac{\ell_1}{b_1}$$

Rearrange.

$$\ell_1 = \frac{b_1 \Delta\ell}{b_2 - b_1}$$

Substitute into equation for S_{F}.

$$\begin{aligned}
S_{\mathrm{F}} &= \pi b_2 \left(\frac{b_1 \Delta\ell}{b_2 - b_1} + \Delta\ell \right) - \pi b_1 \left(\frac{b_1 \Delta\ell}{b_2 - b_1} \right) \\
&= \pi \left(\frac{b_1 b_2}{b_2 - b_1} + \frac{b_2^2 - b_1 b_2}{b_2 - b_1} - \frac{b_1^2}{b_2 - b_1} \right) \Delta\ell \\
&= \pi \left(\frac{b_2^2 - b_1^2}{b_2 - b_1} \right) \Delta\ell \\
&= \pi \left(\frac{(b_2 - b_1)(b_2 + b_1)}{b_2 - b_1} \right) \Delta\ell \\
&= \pi (b_2 + b_1) \Delta\ell
\end{aligned}$$

Use Pythagorean principle.

$$S_{\mathrm{F}} = \pi (b_2 + b_1) \sqrt{\Delta h^2 + (b_2 - b_1)^2}$$

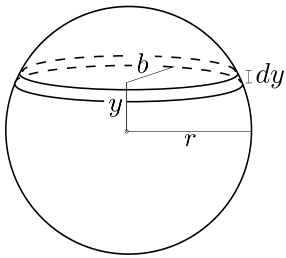
$$S_{\mathrm{F}} = \pi (b_2 + b_1) \sqrt{1 + \left(\frac{b_2 - b_1}{\Delta h} \right)^2} \Delta h$$

If Δh becomes very small ($\Delta h \to \mathrm{d}h$), and $b_2 = b_1 + \mathrm{d}b$,

$$\mathrm{d}S = 2\pi b\sqrt{1 + \left(\frac{db}{dh}\right)^2}\,\mathrm{d}h$$

This is the general equation for the differential area of a surface of revolution. This is usually just memorized, so you could have basically started from here.

Now we return to our sphere. We can relate how far up the section is to how wide the frustum is.



$$b = \sqrt{r^2 - y^2}$$

$$\frac{\mathrm{d}b}{\mathrm{d}h} = \frac{-\,y}{\sqrt{r^2 - y^2}}$$

$$\mathrm{d}h = \mathrm{d}y$$

$$\mathrm{d}S = 2\pi\sqrt{r^2 - y^2}\sqrt{1 + \frac{y^2}{r^2 - y^2}}\,\mathrm{d}y$$

$$\mathrm{d}S = 2\pi\sqrt{r^2 - y^2 + y^2}\,\mathrm{d}y$$

$$\mathrm{d}S = 2\pi r\,\mathrm{d}y$$

Thus, our differential surface area is independent of y, the position of the slice. This means that each tiny slice (of equal thickness) of the spherical potato has the same amount of skin. This can be extended to larger slices quite easily because 100 tiny slices from anywhere on the sphere will have the same amount of skin as 100 other tiny slices.